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On spheroidality and ellipsoidality of the boundaries of numerical ranges of algebraic elementary operators

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ABSTRACT
Geometrical properties of elementary operators form a core area of interest in functional analysis. Determining the shape of the boundary of the numerical ranges of algebraic elementary operators still remains interesting. In this note, we determine the shape of the boundary of the numerical ranges of algebraic elementary operators. In particular, we show that the boundaries conform to spheroidality and ellipsoidality criteria. We also show the relationship between these two shapes.

KEYWORDS
Algebraic elementary operator, Boundary, Numerical range, Spheroidality, Ellipsoidality.

1. Introduction

Several properties of the numerical range (NR) are considered here particularly the boundary for elementary operator (EO). We review related literature on these aspects. In [38] the researchers gave characterizations of NRs in a broader picture and worked on the link between NRs of different types like the essential NR, joint NR, classical NR, joint essential NR, spatial NR among others. In a comparative work in [1], the classical NR was compared with other types of NRs in-terms of their spherical shape. This also forms a significant milestone on the study of NRs as seen in [6]. However, this study did not consider EOs at all not even the special case of algebraic elementary operator (AEO) and therefore it is interesting to determine if a characterization in this perspective gives similar results [5]. Another work in [2] also considered the classical NR of an operator and showed that it is equal to its algebraic NR.

Lemma 1. *([2]) The classical NR of an operator A is equal to its algebraic NR.*

In [2], the author studied and showed that the classical NR of an operator A is equal to its algebraic NR. This shows that the classical NR are entirely in the algebraic NR. This characterization was precisely for Hilbert space (HS) operators. It is not known whether this result applies analogously for AEO. The work of [7] on the classical NR

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of EOs, carried out interesting research on a general setting of Banach algebras (BA). The study was very particular on the classical NR and their properties in Banach algebras as seen in the next result. The location of the points of the classical NR therefore forms an aspect of this study. and the research embarked on the classical NR and the point spectrum (Sp) and their characterizations. The author in [9] discussed the extremal points of the classical NR. The study established that these points are contained in the point Sp of the operator. This applies to HS operators as seen in [10]. It is however not clear whether this assertion applies to EOs or AEOs in particular [11]. This work therefore considers the points of the classical NR of AEO and checks whether they are contained in the point Sp of the AEO.

Theorem 1. *([33]) The classical NR of an operator A is equal to its algebraic NR.*

In [33] the work considered the boundary of a different type of NR that is the joint spatial NR, it was determined that the boundary points belong to subsets that are connected. As observed in [36], the study considered the joint spatial NR and the subsets that connected. The works showed that the boundary points of the joint spatial NR is contained subsets that form the convex hull of the operator. The focus was on HS operators but a consideration was not given for complex operators like the EOs. In particular, no consideration has been given to AEO in this respect [12]. In [8] on a study of NRs of matrix operators and dilations characterized NRs and their NRs for special cases.

Proposition 1. *([8]) C-NRs of square matrices are equal to NRs of their unitary dilations.*

Dilations are maps with very interesting properties and their NRs are sets with good characterizations. In [8], the author described these operators and the study was of NRs of matrix operators and dilations characterized NRs and their NRs for spacial cases. The work determined that for matrix operators which are square, their C-NRs are equal to NRs of their unitary dilations. The work did not however talk about the boundary points of the U-NR. This therefore requires a study for the boundary points of the NRs of these dilations and also for AEOs as illustrated in [15]. The study of [30] worked on NRs of maps that are normal in HSs and characterized EOs induced by them. They established that that for normal maps, convexity of their NRs holds. The study also considered the boundary of the NRs of the normal maps. However, the EOs were not considered in this study even though it is known that that they are induced by normal maps they also have convex NRs. This result is a confirmation of one of the well known results on convexity that the NR is always convex [3]. This characterization holds true for BA setting in general and also in all the special cases that have been considered [26].

Theorem 2. *([24]) The boundary of the NR of a BEO contains extremal points.*

Another consideration on NRs is the case of BEO. The author in [24] considered the NR of the BEO. Regarding closedness, the author of [22] determined that the boundary of NR of a BEO contains extremal points. The BEO is very instrumental since its a map that must be induced by other operators [13]. It is interesting to determine if other examples of EOs like inner derivation (ID), generalized derivation (GD), Jordan elementary operator (JEO) and an elementary operator in general have these properties [14]. In another study on complex spaces, the researchers in [18] studied the shape of the boundary of NRs in complex spaces and considered the

EOs when they are induced by hyponormal operators. With regard to geometrical properties, the shape of the boundary of a NR is a very significant property that has been considered by several researchers [14]. The authors in [18] showed that the boundary of the NR forms a compact spheroid. The compactness property is important since it relays the aspect of closedness [17]. The closedness of the NR implies that the elements in boundary of the NR are limit points and hence the boundary forms a subset that satisfies the compactness criterion for HS [21]. As an analogy in [20], the EOs implemented by hyponormal operators were also studied and similar result arrived at in terms of the shape of the boundary of NRs but the shape was not big enough in size compared to other classes of operators when the real case was given a consideration [23]. This work did not study consider AEOs that are and therefore it is interesting to establish if a characterization in this regard gives similar results or a deviation for AEOs. Recently, The research by [27] considered joint NRs. They considered the shape of the boundary of NRs like the essential NR, joint NR and joint essential NRs and their relatives.

Theorem 3. *([27], Theorem 3.7) The boundary of joint NR of a pair of HS operators is not convex.*

The characterization in [27] considered a pair of HS operators and characterized the joint NR for HS operator. They showed how complex the underlying structures can be when a pair of operators are considered and not individual operators. Emphasis can be given to an AEO which is implemented by this pair to determine if the convexity criterion is not satisfied. Another study as seen in the work of [11] characterized the property of completeness and showed that a research done in a stricter sense for HS operators regarding the completeness of NRs yielded positive results.

Theorem 4. *([11], Theorem 1) Orthogonal sets of maps that satisfy BJO are complete and related to the numerical range of EO.*

Completeness is an important property of sets and operators that are always interesting to characterize [25]. Results in [29] shows a characterization of polaroid maps which is also a very useful area in the theory of operators in terms of certain fundamental properties. This research dwelt on the aspect of completeness and also on the numerical ranges as observed by [28]. It characterized and showed that the NR is complete particularly for HS operators but not AEOs. A lot of studies ([31], [32] and [16] and the references therein) have been done in this respect by considering operators which are normal, normaliod, spectraloid, norm-attaining, hyponormal, (p, q)-hyponormal, norm-attainable among others. The studies however did not characterize on algebraicity of EOs. Even the structural properties EOs that are algebraic were not considered and therefore it is interesting to determine if a characterization in this perspective gives similar results or a deviation for AEOs [36]. To summarize the mathematical background we consider the link between orthogonality and the numerical ranges of maps. Several studies have been carried out in this respect particularly in normed spaces (NS). For instance, the author in [35] studied orthogonality (ORTHO) and Birkhoff-James orthogonality (BJO) and NRs in NS. To finish this section, we consider the linkage between ORTHO and the boundaries of NRs for operators. Several studies have been carried out in this respect particularly in normed spaces (NS). For instance, [35] studied BJO and NRs in NS. The author showed that BJO for two operators is satisfied if zero is contained in the NR of one of the operators.

Theorem 5. *([35], Corollary 4.6). BJO for two operators is satisfied if zero is contained in the NR of any one of the operators.*

As stated in Theorem 5, we observe that there is a link between BJO and NRs in NS. By considering two NS operators, it is noted that when zero is contained in the NR then we can use this element to attain BJO [36]. It is therefore interesting to know the relationship with regard to the boundary of the Nr particularly for AEOs [37]. Another recent study by [41] also sought to determine the relationship between ORTHO via orthogonal sets and NRs. They showed that there is a link between ORTHO and abstract NR by illustrating that one can write the Nr in-terms of BJO.

Proposition 2. *([41], Corollary 1.2). BJO for two operators is satisfied if zero is contained in the abstract NR of any one of the operators.*

The abstract NR is and has been used when establishing the link between ORTH and NRs [40]. In deed, Proposition 2 asserts that the abstract NR can be written via BJO in NS. The study however considered operators in NSs but not EOs. So it is necessary to consider AEOs to identify if this relationship exists for the boundary of NRs of AEOs and BJO of AEOs. In summary, there is no direct relationship between NRs of AEOs and their BJO [29] and so its prudent to embark on a study to determine this relationship.

2. Preliminaries

Definition 1 ([29]). *A mapping G is an EO if its formation is $G(R) = \sum_{i=1}^{n} M_i R M_i$, for all R in an algebra $\mathcal{A}$, where M_i, B_i are fixed in $\mathcal{A}$. The mapping G is said to be algebraic if for some nonzero polynomial p, we have $p(G) = 0$. We have the left multiplication (LMO), right multiplication (RMO), inner derivation (ID), generalized derivation (GD), basic EO (BEO) and Jordan EO (JEO) as examples of EOs.*

Definition 2 ([4]). *Let $\mathcal{H}$ be a Hilbert space. Two elements $x, y \in \mathcal{H}$ are said to be orthogonal denoted by $x \perp y$, if $\langle x, y\rangle = 0$. We say that subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$ are orthogonal written as $\mathcal{A} \perp \mathcal{B}$, if $x \perp y$ for every $x \in \mathcal{A}$ and $y \in \mathcal{B}$.*

3. Research methodology

Techniques and methods which are useful in obtaining results are given in the present chapter. The background knowledge on HS especially on NR and ORTHO are crucial and their properties is of great value to this study. We state and discuss the methods which are instrumental in obtaining results in this study. We describe Fundamental principles. We also consider some known useful results. Lastly, we describe some technical approaches like direct sum decomposition.

3.1. *Fundamental principles*

Theorem 6. ***(Uniform Convergence Criterion*** *(UCC)) Uniform convergence ensures that the functions f_n get arbitrarily close to f simultaneously across the entire domain E, providing a uniform rate of convergence.*

Stronger convergence features than pointwise convergence are guaranteed by the UCC, a key idea in analysis. It makes the interchange of limits and other operations easier by ensuring that the limit function will remain continuous, integrable, and differentiable. The Cauchy criterion is an essential tool in many mathematical analyzes since it offers a useful way to confirm uniform convergence.

Theorem 7. *(**Closed Graph Theorem** (CGT)) It states that if the graph of an operator T is closed then T is bounded and continuous.*

If the graph of a projection operator in a HS is closed, then the boundedness of the projection can be deduced. It guarantees that linear operators between Banach spaces are continuous given appropriate conditions (closed graph), making study and implementation of such operators easier in a variety of academic and physical settings. The CGT can be used to prove the boundedness of operators that are not initially known to be continuous in real-world applications. This is very helpful for different applications in quantum physics, differential equations, and functional analysis.

Theorem 8. *(**Inverse Mapping Theorem**(IMT)) States that every continuously differentiable operator is closed and bounded.*

It is a fundamental concept in multivariable calculus, the Inverse Mapping Theorem sheds light on the local invertibility of differentiable functions. The inverse function's existence and differentiability are made possible by the requirement that the Jacobian be invertible, which guarantees the function's good behaviour close to the point of interest. This theorem helps solve challenging mathematical puzzles in a variety of academic fields and expands our knowledge of the local structure of functions.

Theorem 9. *(**Bouldin's Criterion**) Let U and V be operators on H having closed ranges then UV also has a closed range if and only if the angle between $Ran(V)$ and $Ker(U) \cap (Ker(U) \cap Ran(V))^{\perp}$ is positive.*

Theorem 10. *(**Cowen's Adjoint Formula** (CAF)) Let ϕ be an analytic selfmap of the unit disc D in a Hardy space H^2. Then its conjugate is also an analytic selfmap of on H^{∞}.*

Theorem 11. *(**Trotter Product Formula** (TPF)) The formula states that the exponential of the sum $A+B$ can be approximated by repeatedly applying the exponentials of A and B individually in a product form. As n goes to infinity, this product converges to $e^{t(A+B)}$.*

An effective technique for estimating the exponential of a sum of operators is the TPF. Some known useful results are also considered in this section. These results are useful in characterizing commutativity of operators.

Theorem 12. *(Toeplitz-Haursdorff Theorem (THT)) NR satisfies convexity criterion always.*

The famous THT is a principle that is concerned with the convexity of sets, in particular, the convexity of NR. It states that the NR satisfies convexity criterion always. Convex sets are very important sets and hence its very useful to always establish whether a set is convex or not. This principle is useful in characterization of EOs with regard to its NR.

Theorem 13. *(SMT). For $\sigma(a)$ the joint Sp of a of elements of an algebra $\mathcal{A}$, and for m-tuples $f = (f_1, f_2, \cdots, f_m)$ of non-commuting polynomials in n variable of $\mathcal{A}$. If a is a commuting system of elements, then $\sigma(a)$ is non-empty, and if $f \in p(\mathcal{A}^n)^m$ is a system of polynomials then $\sigma f(a) = f\sigma(a)$.*

Theorem 14. *(OMT) The range of a map is closed if and only if it is an open.*

The OMT has helped us to establish the relationship between the NR and the Sp of EOs implemented by OPs. It helps in establishing the openness of these two crucial sets of EO and in establishing the ling between them

3.2. *Technical approaches*

Tensor product(TP)-This is a technical approach that is useful in tensor analysis of the operators. We consider TP of EOs and check the underlying structures whether they are well defined and if they possess the properties of operators. Then NR and Sp are characterized for the TP of these operators.
Direct sum decomposition(DSD)-This is a technical approach that is useful in analysis of matricial operators. We consider DSD of EOs and check the underlying structures whether they are well defined and if they possess the properties of matricial operators. Then Sp are characterized for the DSD of these matricial operators.
Polar decomposition of operators(PD)-We have used polar decomposition of operators to develop the proofs of some of the results herein p-norms.

3.3. *Known inequalities*

The involvement of some standard identities and inequalities in this study is indispensable. They include: Cauchy-Schwarz-inequality, Cauchy-Buniakowski-Schwarz inequality and triangle inequality in determination of ORTHO.

4. Main results

The first property considered for characterization is the NR of EO induced by OPs. We note that all OPs are in $B_{op}(\mathcal{H})$. We begin by auxiliary results on OPs which are crucial in the sequel. Unless otherwise stated, the AEOs are implemented by orthogonal projections(OP).Our first characterization is given in the proposition below.

Proposition 3. *For P_1, P_2 fixed in $B_{op}(\mathcal{H})$, the convex hull satisfies the subset criterion, that is, $co(W(P_1) \odot W(P_2))^- \subseteq W_0(R_q(P_1, P_2))$, where $q \geq 1$.*

Proof. Let f be a state in $B_{op}(\mathcal{H})$ with $f(1) = 1$. Since $f(R_q(P_1, P_2)) = \sum_{i=1}^{n} \langle P_{1i}x, x\rangle \cdot \langle P_{2i}y, y\rangle \in W_o((R_q(P_1, P_2))$, we get the following strict inequality, $W(P_1) \circ W(P_2) \subset W_0(R_q(P_1, P_2))$, and since $W_0(R_q(P_1, P_2))$ satisfies compactness criterion and THT of convexity criterion, we have that $co(W(P_1) \circ W(P_2))^- \subset W_o(R_q(P_1, P_2))$.

Proposition 4. *Consider a commutative $B_{op}(\mathcal{H})$. For $P \in B_{op}(\mathcal{H})$, We have that $W_0(L_p|l_q) = W_0(R_p|l_q) = W_0(P)$, where l_q is a general Banach space.*

Proof. The proof for the equality requires that we show both inclusions. We first note that the inclusion $W_0(P) \subseteq W_0(L_p|l_q)$, $W_0(P) \subseteq W_0(R_p|l_q)$ follows from Proposition

3. Therefore, $W_0(L_p|l_q) \subseteq W_0(P)$. An analogy gives $W_0(R_p|l_q) \subseteq W_0(P)$.

Proposition 5. *For $P_1, P_2 \in B_{op}(H)$, we have that for $q \geq 1$, the equality $W_o(\delta_{P_1,P_2}|l_q) = W_o(\delta_{P_1,P_2})$ holds.*

Proof. From Proposition 3 and Proposition 4, we get $W_0(P_1) - W_0(P_2) \subseteq W_0(\delta_{P_1,P_2}|l_q) \subseteq W_0(L_p|l_q) - W_0(R_{P_2}|l_q) = W_0(P_1) - W_0(P_2)$.

Lemma 2. *For $P_i, Q_i \in B_{op}(\mathcal{H})$ the following inequality holds, that is, $co[(W_e(P) \circ W(Q)) \cup W(P) \circ W_e(Q))] \subseteq V_e(R_{2,P,Q})$.*

Proof. Consider an element in the essential NR of P, that is, $\lambda \in W_e(P)$. we can have an orthogonal sequence such that $\langle R_{P,Q}(x_n \otimes y), x_n \otimes y\rangle = \sum_{i=1}^{q} \langle P_i x_n, x_n\rangle \cdot \langle Q_i y, y\rangle$. By [154] we have that $\lambda \circ \mu \in V_e(R_{P,Q})$. Hence, $co[(W_e(P) \circ W(Q)) \cup W(P) \circ W_e(Q))] \subseteq V_e(R_{2,P,Q})$.

Theorem 15. *Let P_1, P_2 be a nonnegative self-adjoint operators and $P_1P_2 = P_2P_1$. Then $V_e(P_1P_2) \subseteq V_e(P_1)V_e(P_2)$.*

Proof. We let $\lambda \in V_e(P_1P_2)$. From Lemma 2 we have $x_n \rightharpoonup 0$ and $\lambda = \lim\langle P_1P_2(x_n), x_n\rangle$. Next, we let $y_n = P_1^{\frac{1}{2}} x_n$. If $y_{n_k} = 0$, for some subsequence, then 0 is in both sides of $V_e(P_1P_2) \subseteq V_e(P_1)V_e(P_2)$. Let $y_n \neq 0$, $\forall n$. We take $z_n = \frac{y_n}{\|y_n\|}$. Now $z_n \rightharpoonup 0$ and $\lambda = \lim\langle P_2 z_n, z_n\rangle \cdot \langle P_1 x_n, x_n\rangle$. From [**?**] $\lim\langle P_2 z_n, z_n\rangle \in V_e(P_2)$ and so $\lambda \in V_e(P_1)V_e(P_2)$. Therefore, $V_e(P_1P_2) \subseteq V_e(P_1)V_e(P_2)$.

Corollary 1. *Let $P_1, P_2 \in B_{op}(\mathcal{H})$. Then $V_e(M_{2,P_1,P_2} \subseteq W(P_1)^- W(P_2)^-$ if P_1, P_2 are unitary.*

Proof. By the essentiality of NR we have from [29] that $L_{P_1}R_{P_2} = R_{P_2}L_{P_1}$, $V_e(L_{P_1}) = W(P_1)^-$ and $V_e(R_{P_2}) = W(P_2)^-$. Let P_1, P_2 be unitary. Then from the statement of Corollary 1, $V_e(M_{2,P_1,P_2} \subseteq W(P_1)^- W(P_2)^-$, if P_1, P_2.

Theorem 16. *Consider a non-commutative $B_{op}(\mathcal{H})$. For the mapping $T : B_{op}(\mathcal{H}) \to B_{op}(\mathcal{H})$ defined by $T(X) = \sum_{i=1}^{n} P_i X Q_i, P_i, Q_i \in B_{op}(\mathcal{H})$, we have $W_0(T(X)) = W_0(P_i) - W_0(Q_i)$.*

Proof. From Proposition 3, we obtain that $W(P_i) - W(Q_i) \subset W_0(T(X))$. By closedness of NR it suffices that $(W(P_i) - W(Q_i))^- = W_0(P_i) - W_o(Q_i) \subset W_o(T(X))$. By Proposition 4, $W_0(T(X)) \subset W_0(L_{P_i}) - W_0(R_{Q_i})) = W_0(P_i) - W_0(Q_i)$ which gives the equality.

Proposition 6. *The NR of LMO is equal to the NR of RMO when they are implemented by OPs.*

Proof. Let A_1, A_2, A_3, A_4 be subsets of $B_{op}(\mathcal{H})$ which are convex. If $A_1 \oplus A_2 = A_3 \oplus A_4$, $A_1 \subseteq A_2$ and $A_3 \subseteq A_4$ then LMO that is $L : A_1 \oplus A_2$ and RMO that is $R : A_3 \oplus A_4$ have equal NRs. To prove this we first show that if $A_1 \oplus A_2 \subseteq A_3 \oplus A_4$ then $A_1 \subseteq A_2$ and $A_3 \subseteq A_4$. Consider the following elements: $x_1, x_2 \in A_1$, $y_1, y_2 \in A_2$, $z_1, z_2 \in A_3$ and $w_1, w_2 \in A_4$. By addition we have $x_1 + y_1 = x_2 + y_2$. So we have that $x_1 + y_1 \in A_1 \oplus A_2$. Analogously, we have that $z_1 + w_1 \in A_3 \oplus A_4$. So, $x_1 + y_1 = z_2 + w_2$. Induction gives through sequences $x_n \in A_1$, $y_n \in A_2$, $z_n \in A_3$ and $w_n \in A_4$ that $nx_1 + y_1 = (z_1 +,,, + z_n) + y_{n+1}$ for $n \geq 1$. now since A_3 satisfies compactness criterion and A_1 is closed and bounded then $A_1 \oplus A_2$ and $A_3 \oplus A_4$ are also closed and bounded. Next we show that LMO is equal to RMO. To do this, we first show that $A_1 \oplus A_2 = A_3 \oplus A_4$. Since we have shown that $A_1 \oplus A_2 \subseteq A_3 \oplus A_4$ where $A_1 \subseteq A_2$ and $A_3 \subseteq A_4$ then by [37] the reverse inclusion suffices implying that $A_3 \oplus A_4 \subseteq A_1 \oplus A_2$ where $A_2 \subseteq A_1$ and $A_4 \subseteq A_3$. Therefore, $A_1 \oplus A_2 = A_3 \oplus A_4$. Now let the LMO be given by $L(P) = P_1$. If the number of elements in $A_1 \oplus A_2$ is equal to the number of elements in $A_3 \oplus A_4$ for the RMO, then by DSD, LMO is equal to RMO.

Remark 1. *The NR of LMO induced by P that is an OP in $B_{op}(\mathcal{H})$ is equal to the classical NR of $P \in B_{op}(\mathcal{H})$. Also, The NR of RMO induced by P that is an OP in $B_{op}(\mathcal{H})$ is equal to the classical NR of $P \in B_{op}(\mathcal{H})$.*

Lemma 3. *The NR of AGD restricted to NS is equal to classical NR of individual operators.*

Proof. Let P_1, P_2 be in $B_{op}(\mathcal{H})$. For the GD $\sigma_{P_1,P_2} = P_1P - PP_2$ we have that NR of σ_{P_1,P_2} is equal to NR of P. To see this, we consider Remark 1 and we obtain that the difference between NR of P_1 and NR of P_2 is a strict subset of NR of GD restricted to the class of all OPs which is in turn a strict subset of the difference of the NR of LMO and NR of RMO also both restricted to the class of all OPs which is equal to the Nr of the difference between the NR of P_1 and NR of P_2.

Remark 2. *For BEO, the result in Lemma 3 is not true since BEO has both P_1, P_2 fixed on the left of and the right of P. Moreover, Lemma 3 does not hold for JEO since it is the sum of the BEOs.*

Remark 3. *For the product of NRs we have that the convex hull of the product of classical NR of P_1 and classical NR of P_2 is a strict subset of classical NR of BEO.*

Theorem 17. *For P_1, P_2 fixed in $B_{op}(\mathcal{H})$, the closure of the NR of P_1 times the closure of the NR of P_2 is a subset of the algebraic NR of BEO.*

Proof. Let P_1, P_2 fixed in $B_{op}(\mathcal{H})$ be convexoid. From [67] we have that the algebraic NR of BEO equals the closure of the convex hull of the product of NR of P_1 and NR of P_2. By inclusion criterion in [42], we have that it is a subset of the closure of the NR of P_1 and NR of P_2. This assertion holds from [13] if P_1 and P_2 are convexoid.

Remark 4. *Theorem 17 holds for JEO when $n = 2$.*

Remark 5. *Theorem 17 also holds for an EO in general when the EO is induced by OPs P_i, Q_i fixed in $B_{op}(\mathcal{H})$.*

Up-to this point it is necessary to characterize in terms of the shape of the NRs. We begin with the proposition on GDs in a general set up.

Theorem 18. *The NR of a GD satisfies the spheroidality criterion and hence its boundary.*

Proof. For the GD given by $\mathcal{D}_{A,B}(X) = AX - XB$, for all $X \in \mathcal{B}(\mathcal{H})$, the NR of $\mathcal{D}_{A,B}$, $W(\mathcal{D}_{A,B}) = \{\langle (AX - XB)x, x\rangle \mid X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{H}, \|x\| = 1\}$, is a spheroid. In deed, from the definition of the NR of $\mathcal{D}_{A,B}$ we have $W(\mathcal{D}_{A,B}) = \{\langle (AX - XB)x, x\rangle \mid X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{H}, \|x\| = 1\}$. Carrying out the expansion of the inner product, we obtain $\langle AXx, x\rangle - \langle XBx, x\rangle$. Applying the adjoint property, gives $\langle XBx, x\rangle = \langle Bx, X^*x\rangle$, which can be rewritten as $\langle AXx, x\rangle - \langle Bx, X^*x\rangle$. If A and B are normal operators, their eigenvalues contribute significantly to the shape of $W(\mathcal{D}_{A,B})$. Now from the THT, the NR is satisfies convexity criterion. But $\mathcal{D}_{A,B}$ is linear, so $W(\mathcal{D}_{A,B})$ lies within an elliptical or spheroidal region. For finiteness case where A and B are $n \times n$ matrices, $A = U\Lambda_A U^*, \quad B = V\Lambda_B V^*$, where Λ_A and Λ_B are diagonal matrices. Then, $\mathcal{D}_{A,B}(X) = AX - XB$. The expression of the eigenvalues take the representation $(\lambda_i - \mu_j)x_{ij}$. This show that the NR satisfies a quadratic relation, leading to an spheroid. Hence the NR of GD satisfies the spheroidality criterion and hence its boundary.

Theorem 19. *The NR of a basic EO satisfies the ellipsoidality criterion and hence its boundary.*

Proof. For the BEO denoted by $\mathcal{E}_{A,B}$ and defined by $\mathcal{E}_{A,B}(X) = AXB$, the NR of $\mathcal{E}_{A,B}$, given by: $W(\mathcal{E}_{A,B}) = \{\langle AXBx, x\rangle \mid X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{H}, \|x\| = 1\}$, is spheroidal. To see this, NR of $\mathcal{E}_{A,B}$ is: $W(\mathcal{E}_{A,B}) = \{\langle AXBx, x\rangle \mid X \in \mathcal{B}(\mathcal{H}), x \in \mathcal{H}, \|x\| = 1\}$. From THT the NR of BEO satisfies convexity criterion. Therefore, $W(\mathcal{E}_{A,B})$ is necessarily convex in $\mathcal{C}$. Now, let $A = U\Lambda_A U^*, \quad B = V\Lambda_B V^*$ be their spectral decompositions, where U, V are unitary matrices and Λ_A, Λ_B are diagonal matrices containing eigenvalues. We have $\mathcal{E}_{A,B}(X) = AXB = U\Lambda_A U^* X V \Lambda_B V^*$. This map preserves the quadratic nature and structure of X and gives ellipticity on the NR and so is its boundary.

Remark 6. *The NR of a Jordan EO satisfies both spheroidicity and ellipticity criteria and hence its boundary.*

The link between ORTHO and the boundary of NRs of AEOs is not known. We embark on characterization of AEOs in this respect. We illustrate that abstract NR can be expressed via BJO.

Proposition 7. *For AEOs, T_1 and T_2 on a NS given as $\mathcal{R}$, we have the abstract NR with respect to $\mathcal{R}$ given as $\mathcal{A}(\mathcal{R}, T_1, T_2)$ containing scalars $\}$ such that T_1 is BJO to $T_2 - \}T_1$.*

Proof. Consider $\}$ in $\mathcal{A}(\mathcal{R}, T_1, T_2)$. Then we can choose ψ in the unit sphere of $\mathcal{R}^*$ for which $\psi(T_1) = 1$, and $\psi(T_2)$ equals to $\}$. This implies that $\psi(T_2 - \}T_1) = 0$ meaning T_1 is BJO to $T_2 - \}T_1$. On the other hand, for a scalars $\}$ such that T_1 is BJO to $T_2 - \}T_1$, we must have a unit sphere of $\mathcal{R}^*$ containing ψ such that $\psi(T_2) = 1$ and $\psi(T_2 - \}T_1) = \psi(T_2) - \}\psi(T_1)$. Hence, $\} = \psi(T_2)$, which is contained in the abstract NR $\mathcal{A}(\mathcal{R}, T_1, T_2)$.

The next lemma shows that the abstract NR can be expressed via BJO with regard to convex hull.

Lemma 4. *For AEOs, T_1 and T_2 on a NS given as $\mathcal{R}$, we have the abstract NR with respect to $\mathcal{R}$ given as $\mathcal{A}(\mathcal{R}, T_1, T_2)$ containing scalars $\}$ such that T_1 is BJO to $T_2 - \}T_1$ if $\mathcal{A}(\mathcal{R}, T_1, T_2)$ is equal the convex hull of the set containing T_1 and T_2.*

Proof. By THT, consider $conv\{T_1, T_2\}$ and let ξ be an extremal point of the unit sphere of $\mathcal{R}^*$. By Proposition 7, we have that $\psi(T_2 - \}T_1) = 0$. That is, T_1 is BJO to $T_2 - \}T_1$ if $A(\mathcal{R}_2, T_1, T_2)$ is equal to the $conv\{T_1, T_2\}$ if $\psi(T_1) = 1$ and $\psi(T_2) = \}$.

Remark 7. *In general, we note that T_1 is BJO to $T_2 - \}T_1$ if the zero operator is in* *the abstract* $\mathcal{A}(\mathcal{R}, T_1, T_2)$*.*

Remark 8. *In particular, we also note that* T_1 *is BJO to* T_2 *if the zero operator is in conv*$\{T_1, T_2\}$*.*

Theorem 20. *For AEOs,* T_1 *and* T_2 *on a NS given as* $\mathcal{R}$*, we have the abstract NR with respect to* $\mathcal{R}$ *given as* $\mathcal{A}(\mathcal{R}, T_1, T_2)$ *containing scalars* $\}$ *such that* T_1 *is BJO to* $T_2 - \}T_1$ *if there is* ζ_k *in the unit ball of* $\mathcal{R}^*$ *such that* $\|T_1\zeta_k\| \to \|T_1\|$ *and* $\langle T_1\zeta_k, T_2\zeta_k\rangle$ *tends to zero for every zero operator is in the abstract* $\mathcal{A}(\mathcal{R}, T_1, T_2)$*.*

Proof. By UCC, we have convergence $\|T_1\zeta_k\| \to \|T_1\|$ and $\langle T_1\zeta_k, T_2\zeta_k\rangle$ tends to zero for every zero operator is in the abstract $\mathcal{A}(\mathcal{R}, T_1, T_2)$ as per the statement of the theorem. From Proposition 7, we have that the abstract NR $\mathcal{A}(\mathcal{R}, T_1, T_2)$ containing scalars $\}$ such that T_1 is BJO to $T_2 - \}T_1$. Again by Lemma 4, we know that $\mathcal{A}(\mathcal{R}, T_1, T_2)$ is equal the convex hull of the set containing T_1 and T_2. Finally, by Remark 7 for a generality and Remark 8 for a particular case the proof is complete.

5. Concluding remarks

Geometrical properties of elementary operators forms a core area of interest in functional analysis. Determining the shape of the boundary of the numerical ranges of algebraic elementary operators still remains interesting. In this note, we determined the shape of the boundary of the numerical ranges of algebraic elementary operators. In particular, we have shown that the boundaries conform to spheroidality and ellipsoidality criteria. We also discussed the relationship between these two shapes.

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